Mappings from classical to outer L^p spaces

We aim to estimate maps T like the following.

$f \in L^p(\mathbb{R})$ = classical L^p space

$\varphi : \mathbb{R} \to \mathbb{R}$, smooth $\operatorname{supp}\varphi \subset [-1,1]$

$$(Tf)(x,t) := \int_{\mathbb{R}} \frac{1}{t} \varphi\left(\frac{x-y}{t}\right) f(y)\, dy \quad , (x,t) \in \mathbb{R}^2_+$$

<u>Problem</u>: Estimate Tf in an outer L^p space.

Abstract tool: outer Marcinkiewicz interpolation.

<u>Prop. 3.5</u>: Let $1 \le p_1 < p_2 \le \infty$ and assume boundedness of

$$T : L^{p_1}(Y,\nu) \to L^{p_1,\infty}(X,\sigma,S)$$

and
$$T : \underbrace{L^{p_2}(Y,\nu)}_{\text{strong classical}} \to \underbrace{L^{p_2,\infty}(X,\sigma,S)}_{\text{weak outer}}$$

where $|T(\lambda f)| = |\lambda T(f)|$ and
$$|T(f+g)| \le C(|T(f)| + |T(g)|)$$

Then $T : L^p(Y,\nu) \to \underbrace{L^p(X,\sigma,S)}_{\text{strong outer}}$

if $p_1 < p < p_2$.

<u>Proof</u>: Consider

$$L^p(Y,\nu) \ni f = f_1 + f_2 \in L^{p_1}(Y,\nu) + L^{p_2}(Y,\nu)$$

We claim

$$\mu(S(T(f)) > C\lambda) \le \mu(S(Tf_1) > \lambda) + \mu(S(Tf_2) > \lambda).$$

As on p. 1:4, take $F = F_1 \cup F_2$ with

$\sup_E S((Tf_1) 1_{X \setminus F_1}, E) \le \lambda$, $\mu(F_1) \le \mu(S(Tf_1) > \lambda) + \varepsilon$

$\sup_E S((Tf_2) 1_{X \setminus F_2}, E) \le \lambda$, $\mu(F_2) \le \mu(S(Tf_2) > \lambda) + \varepsilon$

$\Rightarrow \sup_E S(T(f) 1_{X \setminus F}, E) =$

$= \sup_E S(\underbrace{|T(f_1+f_2)|}_{\leq C(|Tf_1|+|Tf_2|)} 1_{X\setminus F}, E)$ (2:2)

$\leq C\big(\sup_E S((Tf_1)\underbrace{1_{X\setminus F}}_{\leq 1_{X\setminus F_1}}, E) + \sup_E S((Tf_2)\underbrace{1_{X\setminus F}}_{\leq 1_{X\setminus F_2}}, E) \big)$

$\leq 2C\lambda$

$\Rightarrow \mu(S(Tf) > 2C\lambda) \leq \mu(F) \leq \mu(F_1) + \mu(F_2) \leq$

$\leq \mu(S(Tf_1) > \lambda) + \mu(S(Tf_2) > \lambda) + \underbrace{2\varepsilon}_{\text{arb.} > 0}$

We obtain from hypothesis

$\mu(S(Tf) > 2C\lambda) \lesssim \frac{1}{\lambda^{p_1}} \int_Y |f_1|^{p_1} d\nu + \frac{1}{\lambda^{p_2}} \int_Y |f_2|^{p_2} d\nu$

$\|Tf\|^p_{L^p(X,\sigma,S)} \approx \int_0^\infty \lambda^{p-1} \mu(S(Tf) > 2C\lambda)\, d\lambda$

$\lesssim \int_Y \Big(\int_0^\infty (|f_1|^{p_1} \lambda^{p-p_1-1}\, d\lambda + |f_2|^{p_2} \lambda^{p-p_2-1}\, d\lambda \Big) d\nu$

Now choose $f_1 = f \cdot 1_{|f|>\lambda}$ $\quad f_2 = f \cdot 1_{|f| \leq \lambda}$

$\Rightarrow \|Tf\|^p_{L^p(X,\sigma,S)} \lesssim \int_Y \Big(|f|^{p_1} \underbrace{\int_0^{|f|} \lambda^{\overbrace{p-p_1-1}^{>-1}} d\lambda}_{\approx |f|^{p-p_1}} + |f|^{p_2} \underbrace{\int_{|f|}^\infty \lambda^{\overbrace{p-p_2-1}^{<-1}} d\lambda}_{\approx |f|^{p-p_2}} \Big) d\nu$

$\approx \int_Y |f|^p d\nu.$

Case $p_2 = \infty$: May assume $\|T\|_{L^\infty(Y,\nu) \to L^\infty(X,\sigma,S)} \leq 1$.

$\Rightarrow \|Tf_2\|_{L^\infty(X,\sigma,S)} \leq \lambda$

$\Rightarrow \mu(S(Tf_2) > \lambda) = \inf\{\mu(F);\ \sup_E \underbrace{S((Tf_2)1_{X\setminus F}, E)}_{\leq \text{cnt} \sup_{X\setminus F} (Tf_2) \leq \lambda \text{ for all } F} \leq \lambda$

$= 0$ (let $F = \emptyset$)

proceed now as for $p_2 < \infty$ //

2:3

The classical Carleson embedding via outer L_p

Assume ν = Carleson measure on $\mathbb{R}^2_+$, i.e.

$$\nu(T(x,t)) \le C \cdot t = C \cdot \sigma(T(x,t))$$

(pre ~~outer~~ measure from p. 1:2)

for all tents $T(x,t)$

Let T be as above p. 2:1.

Classical result:

$\forall p \in (1,\infty]$ $\forall f \in L_p(\mathbb{R}, dx)$

$$\|Tf\|_{L_p(\mathbb{R}^2_+,\nu)} \le C_p \|f\|_{L_p(\mathbb{R})}$$

New proof via outer measures:

(1) Use S_∞ size from p. 1:2.

We have $\|Tf\|_{L_p(\mathbb{R}^2_+,\nu)} = \|Tf\|_{L_p(\mathbb{R}^2_+,\nu,S_\infty)}$ (measure as outer measure.)

Indeed

$$\nu(S_\infty(Tf) > \lambda) = \inf\{\nu(F);\ \underbrace{\sup_E S_\infty(Tf\,1_{X\setminus F}, E)}_{=\sup_{X\setminus F}|Tf|} \le \lambda\}$$

$$= \nu(\{(x,t);\ |Tf| > \lambda\})$$

(2) $\nu \lesssim \sigma \Rightarrow \nu \lesssim \mu \Rightarrow$

$$\|Tf\|_{L_p(\mathbb{R}^2_+,\nu,S_\infty)} \lesssim \|Tf\|_{L_p(\mathbb{R}^2_+,\sigma,S_\infty)}$$

(3) Claim:

$$\|Tf\|_{L_p(\mathbb{R}^2_+,\sigma,S_\infty)} \lesssim \|f\|_{L_p(\mathbb{R})}, \quad \forall p \in (1,\infty]$$

Proof: $p = \infty$:

$$|Tf(x,t)| \le \|\varphi\|_1 \cdot \|f\|_\infty$$

$$\Rightarrow \|Tf\|_{L_\infty(\mathbb{R}^2_+,\sigma,S_\infty)} = \sup_{(x,t)} S_\infty(Tf, T(x,t))$$

$$= \|Tf\|_\infty \lesssim \|f\|_\infty$$

weak L_1:

Need: $\mu(S_\infty(Tf) > \lambda) \lesssim \frac{1}{\lambda} \|f\|_{L_1(\mathbb{R})}$, $\forall \lambda > 0$.

Let $\Omega := \{x;\ Mf(x) > \lambda\} \subset \mathbb{R}$

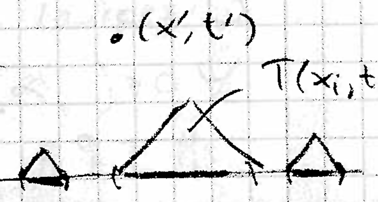
Hardy-Littlewood max. function

$\Omega = \bigcup_i (x_i - t_i, x_i + t_i)$ disjointly

$F := \bigcup_i T(x_i, t_i) \subset \mathbb{R}^2_+$

$(x', t') \notin F \Rightarrow$

$\exists y \in (x' - t', x' + t') : Mf(y) \leq \lambda$



$$\Rightarrow |(Tf)(x', t')| = \left| \int \frac{1}{t'} \varphi\left(\frac{x'-z}{t'}\right) f(z)\, dz \right|$$

$$\lesssim \int_{x'-t'}^{x'+t'} \frac{1}{t'} |f(z)|\, dz \lesssim Mf(y) \leq \lambda$$

$$\Rightarrow \operatorname{outsup}_{F^c} S_\infty(Tf) = \sup_{\text{tents } E} S_\infty(Tf\, 1_{F^c}, E)$$

$$= \|Tf\|_{L_\infty(F^c)} \leq C\lambda$$

$$\Rightarrow \mu(S_\infty(Tf) > C\lambda) = \inf\{\mu(F');\ \operatorname{outsup}_{(F')^c} S_\infty(Tf) \leq C\lambda\}$$

$$\leq \mu(F) = |\Omega| \lesssim \frac{1}{\lambda} \|f\|_1$$

by $M : L_1 \to wL_1$ boundedness

Interpolate to $p \in (1, \infty)$! //

Note: this proof can be done classically by interpolation

$T : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}^2_+)$

$T : L_1(\mathbb{R}) \to L_{1,\infty}(\mathbb{R}^2_+)$

since for S_∞ size

$\nu(\{|Tf| > \lambda\}) = \nu(S_\infty(Tf) > \lambda) \lesssim \mu(S_\infty(Tf) > \lambda)$.

Paraproduct estimates using size S_2

(3.1)

<u>Aim</u>: To estimate the paraproduct trilinear form

$$L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R}) \times L_{p_3}(\mathbb{R}) \longrightarrow \mathbb{C} : \quad =: \Lambda(f_1, f_2, f_3)$$

$$(f_1, f_2, f_3) \longmapsto \int_{\mathbb{R}^2_+} (T_1 f_1) \cdot (T_2 f_2) \cdot (T_3 f_3) \frac{dx\,dt}{t}$$

$$(T_i f)(x,t) := \left(\left(\tfrac{1}{t} \varphi_i\left(\tfrac{\cdot}{t}\right) \right) * f \right)(x)$$

$$\int \varphi_1 = \int \varphi_2 = 0 \qquad \int \varphi_3 \text{ arbitrary.}$$

We use size S_p from p. 1.2.

<u>Prop. 3.6</u>: $\nu = \frac{dx\,dt}{t}$, $\quad f = (T_1 f_1) \cdot (T_2 f_2) \cdot (T_3 f_3)$

Then

$$\left| \int_{\mathbb{R}^2_+} f \, d\nu \right| \lesssim \| f \|_{L_1(\mathbb{R}^2_+, \sigma, S_1)}$$

<u>Proof</u>:

$$\| f \|_{L_1(\mathbb{R}^2_+, \sigma, S_1)} = \int_0^\infty \mu(S_1(f) > \lambda) \, d\lambda$$

$$\approx \sum_k 2^k \underbrace{\mu(S_1(f) > 2^k)}_{= \inf_F \{ \mu(F) ;\ \operatorname{outsup}_{F^c} S_1(f) \leq 2^k \}}$$

Take F_k s.t. $\operatorname{outsup}_{F_k^c} S_1(f) \leq 2^k$ &

$$\mu(F_k) \leq 2 \mu(S_1(f) > 2^k)$$

By defn. of μ:

$$F_k \subset \bigcup_i E_k^i, \qquad \sum_i \sigma(\underbrace{E_k^i}_{\text{tents}}) \leq 2 \mu(F_k)$$

For S_1, we have

$$\forall f : \int_{\underbrace{E}_{\text{tent}}} |f| \, d\nu = S_1(f, E) \, \sigma(E)$$

$$\Rightarrow \left| \int_{\underbrace{\mathbb{R}^2_+}_{= \bigcup_k F_k}} f \, d\nu \right| \leq \sum_k \int_{F_k \setminus \bigcup_{j > k} F_j} |f| \, d\nu \leq$$

$$\leq \sum_k \sum_i \int_{E_k^i \setminus \bigcup_{j>k} F_j} |f| \, d\nu \leq \sum_k 2^k \mu(F_k)$$

$$\underbrace{\leq \underbrace{S_1(f \mathbb{1}_{F_{k+1}^c}, E_k^i)}_{\leq \text{outsup}_{F_{k+1}^c} S(f) \leq 2^{k+1}} \sigma(E_k^i)}$$

$$\lesssim \|f\|_{L_1(\mathbb{R}^2_+, \sigma, S_1)} .$$

Prop. 3.4: (Outer Hölder inequality)

Outer measures: $\mu \leq \mu_1$, $\mu \leq \mu_2$

Sizes: $\forall E \in \mathbb{E}\ \exists E_1 \in \mathbb{E}_1, E_2 \in \mathbb{E}_2\ \forall f_1, f_2 \in \mathcal{B}(X)$:

$$S(f_1 f_2, E) \leq S(f_1, E_1) \cdot S(f_2, E_2)$$

$$\Rightarrow \|f_1 f_2\|_{L^p(\mathbb{R}^2_+, \sigma, S)} \leq 2 \|f_1\|_{L^{p_1}(\mathbb{R}^2_+, \sigma_1, S_1)} \|f_2\|_{L^{p_2}(\mathbb{R}^2_+, \sigma_2, S_2)}$$

when $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

Proof: $j = 1, 2$. $\lambda > 0$ given.

Pick F_j: $\text{outsup}_{F_j^c} S_j(f_j) \leq \lambda^{p/p_j}$

& $\mu_j(F_j) \leq \mu_j(S_j(f_j) > \lambda^{p/p_j}) + \varepsilon$

$F := F_1 \cup F_2$.

$$\Rightarrow \forall E \in \mathbb{E}: S(f_1 f_2 \mathbb{1}_{F^c}, E) \overset{\text{hyp.}}{\leq} S_1(f_1 \overset{\leq \mathbb{1}_{F_1^c}}{\mathbb{1}_{F^c}}, E_1) S_2(f_2 \overset{\leq \mathbb{1}_{F_2^c}}{\mathbb{1}_{F^c}}, E_2)$$

$$\leq \lambda^{p/p_1} \lambda^{p/p_2} = \lambda$$

$$\Rightarrow \mu(S(f_1 f_2) > \lambda) \leq \mu(F) \leq \mu(F_1) + \mu(F_2)$$

$$\leq \mu_1(F_1) + \mu_2(F_2) \leq \mu_1(S_1(f_1) > \lambda^{p/p_1}) + \mu_2(S_2(f_2) > \lambda^{p/p_2}) + 2\varepsilon$$

$$\Rightarrow \int p\lambda^{p-1} \mu(S(f_1 f_2) > \lambda) \, d\lambda \leq \sum_j \int p \lambda^{p-1} \mu_j(S_j(f_j) > \lambda^{p/p_j}) \, d\lambda$$

$$= / \tilde{\lambda}^{p_j} = \lambda^p \Rightarrow p_j \tilde{\lambda}^{p_j - 1} d\tilde{\lambda} = p \lambda^{p-1} d\lambda /$$

$$\leq 2 \quad \text{if} \quad \|f_j\|_{L^{p_j}(X, \sigma_j, S_j)} = 1.$$

Back to paraproduct estimate:

$$|\Lambda(f_1, f_2, f_3)| \underset{\text{prop. 3.6.}}{\lesssim} \| \prod T_j(f_j) \|_{L^1(\mathbb{R}^2_+, \sigma, S_1)}$$

$$\underset{\text{Prop. 3.4. twice}}{\lesssim} \| T_1(f_1) \|_{L^{p_1}(\mathbb{R}^2_+, \sigma, S_2)} \| T_2(f_2) \|_{L^{p_2}(\mathbb{R}^2_+, \sigma, S_2)} \| T_3 f_3 \|_{L^{p_3}(\mathbb{R}^2_+, \sigma, S_\infty)}$$

tent

since $S_1(f_1 f_2 f_3, E) \leq S_2(f_1, E)\, S_2(f_2, E)\, S_\infty(f_3, E)$

classical Hölder.

We obtain

$$\boxed{|\Lambda(f_1, f_2, f_3)| \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3} \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}, \quad 1 < p_j \leq \infty}$$

from

<u>Thm 4.1</u>:

$$\|Tf\|_{L^p(\mathbb{R}^2_+, \sigma, S_2)} \lesssim \|f\|_{L^p(\mathbb{R})}, \quad 1 < p \leq \infty$$

if $Tf(x,t) = \frac{1}{t}\varphi\left(\frac{\cdot}{t}\right) * f(x)$ and $\underline{\int \varphi = 0}$.

<u>Proof</u>: Interpolation $1, \infty \to p$ as for S_∞ above.

<u>$p = \infty$</u>: Have

$$\iint_{\mathbb{R}^2_+} |Tg|^2 \frac{dx\,dt}{t} \approx \int_{\mathbb{R}} |g|^2\, dx$$

(Calderón's reprod. formula = Plancherel & Fubini & $\int \varphi = 0$)

We localize to a tent $T(x,s)$:

$$g = f \cdot 1_{(x-3s,\, x+3s)}$$

$\operatorname{supp} \varphi \in [-1, 1] \Rightarrow Tg = Tf$ on $T(x,s)$.

$$\Rightarrow S_2(Tf, T(x,s))^2 = \frac{1}{s} \iint_{T(x,s)} |Tg|^2 \frac{dx\,dt}{t}$$

$$\lesssim \frac{1}{s} \int |g|^2 \leq \|f\|_\infty^2, \quad \forall (x,s).$$

weak L_1:

Need $\mu(S_2(Tf) > \lambda) \lesssim \frac{1}{\lambda}\|f\|_{L_1}$

Calderón-Zygmund decomposition at λ:

$f = g + \sum b_i$

$\|g\|_\infty \le \lambda$, supp $b_i \subset (x_i - s_i, x_i + s_i)$, $\int b_i = 0$

L_∞ estimate $\Rightarrow$ $\text{outsup}_{\mathbb{R}^2_+} S_2(Tg) \lesssim \lambda$

$F := \bigcup T(x_i, 3s_i)$, $b := \sum b_i$

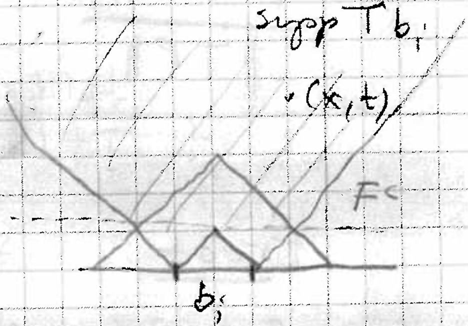
Need: $\text{outsup}_{F^c} S_2(Tb) \lesssim \lambda$

/If so, then

$$\mu(S_2(Tf) > c\lambda) \le \underbrace{\mu(S_2(Tg) > d\lambda)}_{=0} + \underbrace{\mu(S_2(Tb) > d\lambda)}_{\le \mu(F)}$$

M boundedness

$$\lesssim \sum s_i \lesssim \frac{1}{\lambda}\|f\|_1$$ /



Let $B_j(x) = \int_{-\infty}^{x} b_j$

Integration by parts $\Rightarrow$

$$T(b_j)(x,t) = \frac{1}{t} B_j * \left(\frac{1}{t}\varphi'\left(\frac{\cdot}{t}\right)\right)$$

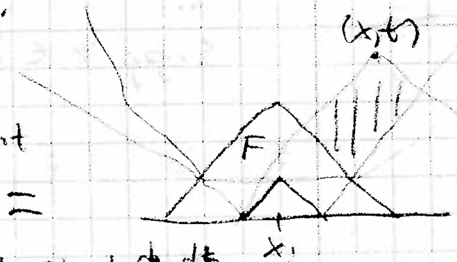
If $(x,t) \notin F$, then $Tb_j(x,t) = 0$ if $t < s_j$

$$\Rightarrow |Tb(x,t)| \le \underbrace{\Big\|\frac{1}{t}\sum_{s_i \le t} B_i\Big\|_\infty}_{\lesssim \frac{1}{t}\sup_i \|b_i\|_1 \lesssim \lambda} \underbrace{\|\varphi'\|_1}_{\lesssim 1} \lesssim \lambda$$

(CZ cst.

$\Rightarrow$ $S_\infty(Tb \cdot 1_{F^c}, E) \lesssim \lambda$

(any tent.



- Estimate of $t S_1(Tb_i 1_{F^c}, T(x,t)) =$

$$= \iint_{T(x,t)\setminus F} |Tb_i| \frac{dy\,ds}{s} \le \int_{s \ge s_i} \int_{|y-x_i| \lesssim t} |Tb_i| \frac{dy\,ds}{s}$$

$$\lesssim \int_{s_i}^{\infty} \int_{|y-x_i| \lesssim t} \|B_i\|_1 \|\varphi'\|_\infty \frac{dy\,ds}{s^3}$$

$\le \|B_i\|_1 \cdot \frac{1}{s_i} \le \|b_i\|_1 \le \lambda \cdot s_i$

$\uparrow$ / $\|B_i\|_1 \le s_i \cdot \|B_i\|_\infty \le s_i \|b_i\|_1$ /

$\Rightarrow S_1(Tb \cdot 1_{F^c}, T(x,t)) \le \sum_i S_1(Tb \cdot 1_{F^c}, T(x,t))$

$\le \frac{1}{t}\lambda \sum_{i: T(x_i, s_i) \cap T(x,t) \neq \phi} s_i \le \lambda$

Classical $L_2 \le L_1 L_\infty$ estimate

$\Rightarrow \operatorname{outsup}_{F^c} S_2(Tb) \lesssim \lambda$ //

On the relation SIOs $\longleftrightarrow$ paraproducts

A singular integral operator T can be decomposed (Coifman–Beylkin–Rokhlin)

$T = \underbrace{\Pi^+_{T(1)} + \Pi^-_{T^*(1)} + \Pi^0_m}_{\text{3 different paraproduct operators corresponding to trilinear paraproduct forms } \Lambda.} + \tilde{T}$

$\int f \cdot \Pi^+_{T(1)} g = \Lambda_1(f, T(1), g)$

$\int f \cdot \Pi^-_{T^*(1)} g = \Lambda_2(g, T^*(1), f)$

$\int f \cdot \Pi^0_m g = \Lambda_3(f, g, m)$

$\tilde{T}$ are "tails" which are bounded only by kernel estimates, as in Thm 4.4.